

A Laurent series proof of the Habsieger-Kadell q -Morris identity

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Abstract

We give a Laurent series proof of the Habsieger-Kadell q -Morris identity, which is a common generalization of the q -Morris Identity and the Aomoto constant term identity.

Keywords: Laurent series; constant term identities; q -Morris identity; q -Dyson identity; Selberg integral

1 Introduction

This paper is closely related to the well-known Dyson's ex-conjecture. The conjecture was made by Freeman Dyson in 1962 when studying Statistical theory of energy levels of complex systems [5].

Theorem 1.1. *For nonnegative integers a_0, \dots, a_n ,*

$$\text{CT}_{\mathbf{x}} \prod_{0 \leq i < j \leq n} \left(1 - \frac{x_i}{x_j}\right)^{a_i} \left(1 - \frac{x_j}{x_i}\right)^{a_j} = \frac{(a_0 + \dots + a_n)!}{a_0! a_1! \dots a_n!}, \quad (1.1)$$

where $\text{CT}_{\mathbf{x}} f(\mathbf{x})$ means to take the constant term in the Laurent expansion of $f(\mathbf{x})$ in the powers of x_0, x_1, \dots, x_n .

Dyson's ex-conjecture has been proved by many authors using different methods. See, e.g., [6, 8, 9, 20, 21]. Many variations of Dyson's ex-conjecture have been found, such as the famous Macdonald constant term conjecture [4, 16]. Some of them are still not solved.

The q -analogous of the Dyson conjecture was made by Andrews [1] in 1975.

Theorem 1.2 (Zeilberger-Bressoud). *For nonnegative integers a_0, a_1, \dots, a_n ,*

$$\text{CT}_{\mathbf{x}} \prod_{0 \leq i < j \leq n} \left(\frac{x_i}{x_j}\right)_{a_i} \left(\frac{x_j}{x_i}q\right)_{a_j} = \frac{(q)_{a_0+a_1+\dots+a_n}}{(q)_{a_0}(q)_{a_1} \dots (q)_{a_n}},$$

where $(z)_m := \frac{(z;q)_{\infty}}{(zq^m;q)_{\infty}} = (1-z)(1-zq) \dots (1-zq^{m-1})$.

Almost all methods for Dyson's ex-conjecture fail to extend for the q version. Up to now, only three different methods succeeded: the combinatorial proof in [22], the short proof in [7] using iterated Laurent series, and the one page proof in [13] using the combinatorial nullstellensatz. The methods apply to some constant terms of similar type.

In this paper we study the Habsieger-Kadell q -Morris identity, an important variation of the equal parameter case of the q -Dyson theorem. Let

$$A_q(x_0, x_1, \dots, x_n; a, b, m, l, k) = \prod_{i=1}^n \left(\frac{q^{\chi(i \leq m)} x_0}{x_i} \right)_{a - \chi(i \leq m)} \left(\frac{q^{\chi(i > m)} x_i}{x_0} \right)_{b + \chi(i \leq m) + \chi(i \geq n - l + 1)} \prod_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} \right)_k, \quad (1.2)$$

where the expression $\chi(S)$ is 1 if the statement S is true, and 0 otherwise.

Theorem 1.3 (Habsieger-Kadell). *For nonnegative integers a, b, m, l and k ,*

$$\text{CT}_x A_q(x_0, x_1, \dots, x_n; a, b, m, l, k) = M_n(a, b, k, m, l; q), \quad (1.3)$$

where

$$M_n(a, b, k, m, l; q) = \prod_{i=0}^{n-1} \frac{(q)_{a+b+ik+\chi(i \geq n-l)} (q)_{(i+1)k}}{(q)_{a+ik-\chi(i < m)} (q)_{b+ik+\chi(i \geq n-m-l)} (q)_k}. \quad (1.4)$$

The $m = l = 0$, $q = 1$ case of the result is the Morris identity, which is equivalent to the well-known Selberg integral [18]. In his thesis [17] Morris established the identity and conjectured the q -analogous identity. The q -Morris identity, or the $m = l = 0$ case, was proved by Habsieger [10] and later by Zeilberger [23]. The $q = 1$ case of the result, called the Aomoto identity, was constructed by Aomoto [3]. By extending Aomoto's method Kadell [11] constructed the above general case, in the same year of Habsieger's proof.

Our approach is by extending the proof of the Aomoto identity in [6]. The basic idea is to regard both sides of (1.3) as polynomials in q^a of degree at most $d = nb + m + l$. Then to show the equality of the two polynomials, it is sufficient to show that they are equal at $d + 1$ points. The equality at the d vanishing points are not hard to handle by the techniques in [7, 15]. But in this approach, we have to deal with two problems: i) the multiple roots problem for small k ; ii) the $d + 1$ -st suitable point is hard to find. We handle the former problem by a rationality result of Stembridge, and the latter problem by a hard searching process.

We present the major steps of our proof in Section 2. The steps are expanded by the rationality result in Section 3, by the proof of the vanishing lemma in Section 4, and by the proof for the extra point in Section 5.

While we were finishing the presented work, the one page proof of the q -Dyson theorem was published. Moreover, Károlyi and Nagy [14] found a short proof of Theorem 1.3 using the combinatorial nullstellensatz. The two approaches are different but have some connections.

2 Proof of the Habsieger-Kadell q -Morris identity

Following notations in the introduction, we may assume that $0 \leq m, l < n$ by the following argument. If $m \geq n$ then

$$A_q(x_0, \dots, x_n; a, b, m, l, k) = \prod_{i=1}^n \left(\frac{qx_0}{x_i} \right)_{a-1} \left(\frac{x_i}{x_0} \right)_{b+1+\chi(i \geq n-l+1)} \prod_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} \right)_k,$$

which is just $A_q(x_0q, \dots, x_n; a-1, b+1, 0, l, k)$. Then by substituting x_0 by x_0/q , we need to verify that $M_n(a, b, m, l, k) = M_n(a-1, b+1, 0, l, k)$ which is easy. The case $l \geq n$ is similar. We observe that $A_q(x_0, \dots, x_n; a, b, m, l, k)$ can be rewritten as $A_q(x_0, \dots, x_n; a, b+1, m, 0, k)$ when $l \geq n$.

Let us rewrite $M(q^a, q^k) = M_n(a, b, k, m, l; q)$ as

$$\begin{aligned} M(q^a, q^k) &= \frac{(q)_{nk}}{(q)_k^n} \cdot \prod_{i=0}^{m-1} (1 - q^{a+ik}) \cdot \prod_{i=n-l}^{n-1} (1 - q^{a+ik+b+1}) \\ &\quad \cdot \prod_{i=0}^{n-1} \frac{(1 - q^{a+ik+1})(1 - q^{a+ik+2}) \cdots (1 - q^{a+ik+b})(q)_{ik}}{(q)_{b+ik+\chi(i \geq n-m-l)}}. \end{aligned} \quad (2.1)$$

We have the following characterization.

Lemma 2.1. *For fixed $b, n \in \mathbb{N}$ and $0 \leq m, l < n$, $M(q^a, q^k)$ is uniquely determined by the following three properties.*

1. $M(q^a, q^k)(q)_k^n / (q)_{nk}$ is a polynomial in q^a of degree $nb + m + l$ with coefficients rational function in q^k .
2. For any $k > b + 1$, $M(q^{-\xi}, q^k) = 0$ if ξ belongs to one of the following three sets:

$$\begin{aligned} D_1 &= \{0, k, \dots, (m-1)k\}; \\ D_2 &= \{(n-l)k + b + 1, (n-l+1)k + b + 1, \dots, (n-1)k + b + 1\}; \\ D_3 &= \{ik + 1, ik + 2, \dots, ik + b \mid i = 0, \dots, n-1\}. \end{aligned} \quad (2.2)$$

3. For any $k > b+1$ we have $M(q^{-(n-l-1)k-b-1}, q^k) = M_n(-(n-l-1)k-b-1, b, k, m, l; q)$.

Proof. Assume $M'(q^a, q^k)$ also satisfies the above three properties. Then for every $\xi \in D_i, i = 1, 2, 3$ or $\xi = (n-l-1)k + b + 1$

$$M(q^{-\xi}, q^k)(q)_k^n / (q)_{nk} = M'(q^{-\xi}, q^k)(q)_k^n / (q)_{nk}, \quad \text{for all } k > b + 1.$$

Since both sides are rational functions in q^k and they agree at infinitely many points, they are identical as rational functions.

Now as polynomials in q^a , with coefficients rational functions in q^k , $M(q^a, q^k)(q)_k^n / (q)_{nk}$ agrees with $M'(q^a, q^k)(q)_k^n / (q)_{nk}$ at $nb + m + l + 1$ distinct ξ 's above, they must be equal to each other. \square

Note that the condition $k > b + 1$ can not be dropped, since D_3 has duplicate elements when $k \leq b + 1$.

Denote by $M'_n(a, b, k, m, l; q)$ the left-hand-side of (1.3). Then Theorem 1.3 will follow by induction on n if we can show the following three lemmas, whose proofs will be given in later sections.

Lemma 2.2. *For fixed $b, n \in \mathbb{N}$ and $0 \leq m, l < n$, $M'_n(a, b, k, m, l; q)(q)_k^n / (q)_{nk}$ is a polynomial in q^a of degree at most $nb + m + l$ with coefficients rational functions in q^k .*

Since $M'_n(a, b, k, m, l; q)$ is a polynomial in q^a , the definition of a can be extended for all integers, in particular for negative integers a .

Lemma 2.3 (Vanishing Lemma). *For fixed $b, n \in \mathbb{N}$, and $0 \leq m, l < n$, and $k > b + 1$, $M'_n(-h, b, k, m, l; q)$ vanishes when h equals one of the values in (2.2).*

Remark 2.4. *At this stage we can already claim the truth of Theorem 1.3 for $m = 0$. In this case D_1 is empty, so that we can choose $a = 0$ as the extra point. Then $M'_n(0, b, k, 0, l; q)$ reduces to $M'_n(0, 0, k, 0, 0; q)$, and the equal parameter case of the q -Dyson theorem applies.*

Lemma 2.5. *Assume Theorem 1.3 holds for smaller n . For fixed $b, n \in \mathbb{N}$, and $0 \leq m, l < n$, and $k > b + 1$, we have $M'_n(-h, b, k, m, l; q) = M_n(-h, b, k, m, l; q)$ when $h = (n - l - 1)k + b + 1$.*

The extra point in the above lemma is found through a hard searching process. It is a surprise for this special h : the constant term $M'_n(-h, b, k, m, l; q)$ reduces to a single constant term that can be evaluated by Remark 2.4 or the hypothesis.

3 The polynomial-rational characterization

To prove Lemma 2.2, we need the the following rationality result, which is implicitly due to J.R. Stembridge [19], as can be seen from the proof. The $q = 1$ case of this result appeared in [6].

Proposition 3.1. *For any $n \in \mathbb{N}$ and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ with $\sum_{1 \leq i \leq n} \alpha_i = 0$, we have*

$$[x^\alpha] \prod_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k = \frac{(q)_{nk}}{(q)_k^n} \cdot R_n(q^k; q). \quad (3.1)$$

where $R_n(q^k; q)$ is a rational function in q^k and q , and $[x^\alpha]$ refers to take the coefficient of $x_1^{\alpha_1} \dots x_n^{\alpha_n}$ in the polynomial.

Proof. In [19, Equation 44] Stembridge gave the following equation (set $z = q^k$)

$$[x^\alpha] \prod_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k = \frac{1}{(q; q)_{k-1}^n} \sum_S \pm (-q^k)^{|S|} C^n[S](q^k, q), \quad (3.2)$$

where S sums over finite elements and $C^n[S](q^k, q)$ is a formal power series in q^k .

By [19, Corollary 3.3] we know that

$$C^n[\emptyset](q^k, q) = \frac{(q)_{nk}}{(1 - q^k)(1 - q^{2k}) \cdots (1 - q^{nk})}. \quad (3.3)$$

In [19, Page 334, Line 33] Stembridge stated that $C^n[\lambda](q^k, q)$ is of the form $f_\lambda(q^k, q) \cdot C^n[\emptyset](q^k, q)$ for some rational function f_λ . Therefore, combining with (3.2) and (3.3) we get

$$[x^\alpha] \prod_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k = \frac{(q)_{nk}}{(q)_{k-1}^n (1 - q^k) \cdots (1 - q^{nk})} \sum_S \pm (-q^k)^{|S|} f_S(q^k, q). \quad (3.4)$$

The desired rational function is then given by

$$R_n(q^k, q) = \frac{(1 - q^k)^n}{(1 - q^k)(1 - q^{2k}) \cdots (1 - q^{nk})} \sum_S \pm (-q^k)^{|S|} f_S(q^k, q).$$

□

Proof of Lemma 2.2. When regarded as Laurent series in x_0 , the equality

$$\left(\frac{x_0}{x_i} \right)_a \left(\frac{x_i}{x_0} q \right)_b = q^{\binom{b+1}{2}} \left(-\frac{x_i}{x_0} \right)^b \left(\frac{x_0}{x_i} q^{-b} \right)_{a+b}$$

can be easily shown to hold for all integers a . Rewrite $M'_n(a, b, k, m, l; q)$ as

$$\text{CT}_x \prod_{i=1}^n q^{\binom{b_i^*+1}{2}} \left(-\frac{x_i}{x_0} \right)^{b_i^* + \chi(i \leq m)} \left(\frac{x_0}{x_i} q^{-b_i^*} \right)_{a+b_i^*} \prod_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k, \quad (3.5)$$

where $b_i^* = b + \chi(i \geq n - l + 1)$.

The well-known q -binomial theorem [2, Theorem 2.1] is the identity

$$\frac{(bz)_\infty}{(z)_\infty} = \sum_{k=0}^{\infty} \frac{(b)_k}{(q)_k} z^k. \quad (3.6)$$

Setting $z = uq^n$ and $b = q^{-n}$ in (3.6), we obtain

$$(u)_n = \frac{(u)_\infty}{(uq^n)_\infty} = \sum_{k=0}^{\infty} q^{k(k-1)/2} \begin{bmatrix} n \\ k \end{bmatrix} (-u)^k \quad (3.7)$$

for all integers n , where $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(q)_n}{(q)_k (q)_{n-k}}$ is the q -binomial coefficient.

Using (3.7), we see that for $1 \leq i \leq n$,

$$q^{\binom{b_i^*+1}{2}} \left(-\frac{x_i}{x_0} \right)^{b_i^* + \chi(i \leq m)} \left(\frac{x_0}{x_i} q^{-b_i^*} \right)_{a+b_i^*} = \sum_{k_i \geq 0} C(k_i) \begin{bmatrix} a + b_i^* \\ k_i \end{bmatrix} x_0^{k_i - b_i^* - \chi(i \leq m)} x_j^{b_i^* + \chi(i \leq m) - k_i},$$

where $C(k_i) = (-1)^{k_i + b_i^* + \chi(i \leq m)} q^{\binom{b_i^*+1}{2} + \binom{k_i}{2} - k_i b_i^*}$.

Expanding the product in (3.5) and taking constant term in x_0 , we see that, by Proposition 3.1, $M'_n(a, b, k, m, l; q)$ becomes

$$\sum_{\mathbf{k}} \prod_{i=1}^n \begin{bmatrix} a + b_i^* \\ k_i \end{bmatrix}_{x_1, \dots, x_n}^{\text{CT}} L(x_1, \dots, x_n; \mathbf{k}) = \frac{(q)_{nk}}{(q)_k^n} \sum_{\mathbf{k}} \prod_{i=1}^n \begin{bmatrix} a + b_i^* \\ k_i \end{bmatrix} R(q^k, q; \mathbf{k}), \quad (3.8)$$

for some rational functions $R(q^k, q; \mathbf{k})$ in q^k and q , where

$$\begin{aligned} L(x_1, \dots, x_n; \mathbf{k}) = & q^{(n-l)\binom{b+1}{2} + l\binom{b+2}{2} + \sum_{i=1}^n \binom{k_i}{2} - b \sum_{i=1}^{n-l} k_i - (b+1) \sum_{i=n-l+1}^n k_i} \\ & \cdot \prod_{i=1}^m x_i^{b_i^* + 1 - k_i} \prod_{i=m+1}^n x_i^{b_i^* - k_i} \prod_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k \end{aligned} \quad (3.9)$$

is a Laurent polynomial in x_1, \dots, x_n independent of a and the sum ranges over all sequences $\mathbf{k} = (k_1, \dots, k_n)$ of nonnegative integers satisfying $k_1 + k_2 + \dots + k_n = nb + m + l$. Since $\begin{bmatrix} a + b_i^* \\ k_i \end{bmatrix}$ is a polynomial in q^a of degree k_i , each summand in (3.8) is a polynomial in q^a of degree at most $k_1 + k_2 + \dots + k_n = nb + m + l$, and so is the sum.

The coefficients of $M'_n(a, b, k, m, l; q)(q)_k^n / (q)_{nk}$ in q^a are clearly rational functions in q^k . \square

4 Proof of the vanishing lemma

We will follow notations in [7, 15], where different versions of the vanishing lemma were proposed for dealing with q -Dyson related constant terms. The new vanishing lemma will be handled by the same idea but we have to carry out the details. We will include some basic ingredients for readers' convenience.

In this section, we let $K = \mathbb{C}(q)$, and assume that all series are in the field of iterated Laurent series $K((x_n))((x_{n-1})) \cdots ((x_0))$. The reason for choosing $K((x_n))((x_{n-1})) \cdots ((x_0))$ as a working field has been explained in [7].

We emphasize that the field of rational functions is a subfield of $K((x_n))((x_{n-1})) \cdots ((x_0))$, so that every rational function is identified with its unique iterated Laurent series expansion. The series expansions of $1/(1 - q^k x_i/x_j)$ will be especially important.

$$\begin{aligned} \frac{1}{1 - q^k x_i/x_j} &= \sum_{l=0}^{\infty} q^{kl} x_i^l x_j^{-l}, \text{ if } i < j, \\ \frac{1}{1 - q^k x_i/x_j} &= \frac{1}{-q^k x_i/x_j (1 - q^{-k} x_j/x_i)} = \sum_{l=0}^{\infty} -q^{-k(l+1)} x_i^{-l-1} x_j^{l+1}, \text{ if } i > j. \end{aligned}$$

The constant term of the series $F(\mathbf{x})$ in x_i , denoted by $\text{CT}_{x_i} F(\mathbf{x})$, is defined to be the sum of those terms in $F(\mathbf{x})$ that are free of x_i . It follows that

$$\text{CT}_{x_i} \frac{1}{1 - q^k x_i/x_j} = \begin{cases} 1, & \text{if } i < j, \\ 0, & \text{if } i > j. \end{cases} \quad (4.1)$$

We shall call the monomial $M = q^k x_i / x_j$ *small* if $i < j$ and *large* if $i > j$. Thus the constant term in x_i of $1/(1 - M)$ is 1 if M is small and 0 if M is large.

Constant term operators defined in this way has the important commutativity property:

$$\text{CT}_{x_i} \text{CT}_{x_j} F(\mathbf{x}) = \text{CT}_{x_j} \text{CT}_{x_i} F(\mathbf{x}).$$

The *degree* of a rational function of x is the degree in x of the numerator minus the degree in x of the denominator. For example, if $i \neq j$ then the degree of $1 - x_j/x_i = (x_i - x_j)/x_i$ is 0 in x_i and 1 in x_j . A rational function is called *proper* (resp. *almost proper*) in x if its degree in x is negative (resp. zero).

Let

$$F = \frac{p(x_k)}{x_k^d \prod_{i=1}^m (1 - x_k/\alpha_i)} \quad (4.2)$$

be a rational function of x_k , where $p(x_k)$ is a polynomial in x_k , and the α_i are distinct monomials, each of the form $x_t q^s$. Then the partial fraction decomposition of F with respect to x_k has the following form:

$$F = p_0(x_k) + \frac{p_1(x_k)}{x_k^d} + \sum_{j=1}^m \frac{1}{1 - x_k/\alpha_j} \left(\frac{p(x_k)}{x_k^d \prod_{i=1, i \neq j}^m (1 - x_k/\alpha_i)} \right) \Big|_{x_k=\alpha_j}, \quad (4.3)$$

where $p_0(x_k)$ is a polynomial in x_k , and $p_1(x_k)$ is a polynomial in x_k of degree less than d .

The following lemma has appeared in [15].

Lemma 4.1. *Let F be as in (4.2) and (4.3). Then*

$$\text{CT}_{x_k} F = p_0(0) + \sum_j (F(1 - x_k/\alpha_j)) \Big|_{x_k=\alpha_j}, \quad (4.4)$$

where the sum ranges over all j such that x_k/α_j is small. In particular, if F is proper in x_k , then $p_0(x_k) = 0$; if F is almost proper in x_k , then $p_0(x_k) = (-1)^m \prod_{i=1}^m \alpha_i \text{LC}_{x_k} p(x_k)$, where LC_{x_k} means to take the leading coefficient with respect to x_k .

The following lemma plays an important role in our argument.

Lemma 4.2. *Let k , b and s be nonnegative integers. Then for any k_1, \dots, k_s with $0 \leq k_i \leq (s-1)k + b + 1$ for all i , either $0 \leq k_i \leq b$ for some i , or $1 - k \leq k_j - k_i \leq k$ for some $i < j$, except only when $k_i = (s-i)k + b + 1$ for $i = 1, \dots, s$.*

Proof. Assume k_1, \dots, k_s to satisfy that for all i , $b < k_i \leq (s-1)k + b + 1$, and for all $i < j$, either $k_j - k_i > k$ or $k_j - k_i \leq -k$. Then we need to show that $k_i = (s-i)k + b + 1$ for $i = 1, \dots, s$.

Divide $[b+1, (s-1)k + b + 1]$ into the following $s-1$ parts: $[b+1, k + b + 1]$ and $[(i-1)k + b + 2, ik + b + 1]$, for $i = 2, \dots, s-1$. By the assumption that $b < k_i \leq (s-1)k + b + 1$ for all the s k_i 's, it follows that at least two of the k_i 's have to be in a common range, let

the two be k_{t_1} and k_{t_2} and assume $t_2 < t_1$. It follows that k_{t_1} and k_{t_2} have to be in $[b+1, k+b+1]$ otherwise if $k_{t_1}, k_{t_2} \in [(i-1)k+b+2, ik+b+1]$ for some $i \in [2, s-1]$ then we have $1-k \leq k_{t_1} - k_{t_2} \leq k$ and it contradicts to our assumption. Furthermore, we have $k_{t_1} = b+1, k_{t_2} = k+b+1$ and the remaind $s-2$ k_i 's must be distributed to the $s-2$ ranges $[(i-1)k+b+2, ik+b+1]$ for $i = 2, \dots, s-1$ averagely, otherwise it contradicts to the assumption. Let $k_{t_3} \in [k+b+2, 2k+b+1]$. Then $k_{t_3} = 2k+b+1$ and $t_3 < t_2$ otherwise $1-k \leq k_{t_2} - k_{t_3} \leq k$. Let $k_{t_4} \in [2k+b+1, 3k+b+1]$. Then $k_{t_4} = 3k+b+1$ and $t_3 < t_4$ otherwise $1-k \leq k_{t_3} - k_{t_4} \leq k$. Following this discussion, we have $k_{t_i} = (i-1)k+b+1$ for $i = 1, \dots, s$ and $t_s < \dots < t_1$. Thus we have $t_i = (s-i+1)$ and $k_i = (s-i)k+b+1$ for $i = 1, \dots, s$. \square

Let

$$Q(h) = \prod_{i=1}^m \frac{\binom{x_i/x_0}{b+1+\chi(i \geq n-l+1)}}{(1-x_0/x_i) \cdots (1-x_0/(x_i q^h))} \prod_{i=m+1}^n \frac{\binom{x_i q/x_0}{b+\chi(i \geq n-l+1)}}{(1-x_0/(x_i q)) \cdots (1-x_0/(x_i q^h))} \cdot \prod_{1 \leq i < j \leq n} \left(\frac{x_i}{x_j}\right)_k \left(\frac{x_j}{x_i} q\right)_k. \quad (4.5)$$

By the proof of Lemma 2.2 in Section 3, we have

$$M'_n(-h, b, k, m, l; q) = \text{CT}_{\mathbf{x}} Q(h).$$

The vanishing lemma says that $\text{CT}_{\mathbf{x}} Q(h) = 0$ for every h in (2.2).

We attack the vanishing lemma by repeated application of Lemma 4.1. This will give a big sum of terms, each will be detected to be 0 by Lemma 4.2. This is better summarized in the following Lemma 4.3. To state the lemma, we need more notations.

For any rational function F of x_0, x_1, \dots, x_n , and for sequences of integers $k = (k_1, k_2, \dots, k_s)$ and $r = (r_1, r_2, \dots, r_s)$ let $E_{\mathbf{r}, \mathbf{k}} F$ be the result of replacing x_{r_i} in F with $x_{r_s} q^{k_s - k_i}$ for $i = 0, 1, \dots, s-1$, where we set $r_0 = k_0 = 0$. Then for $0 < r_1 < r_2 < \dots < r_s \leq n$ and $0 \leq k_i \leq h$, we define

$$Q(h \mid \mathbf{r}; \mathbf{k}) = Q(h \mid r_1, \dots, r_s; k_1, \dots, k_s) = E_{\mathbf{r}, \mathbf{k}} \left[Q(h) \prod_{i=1}^s \left(1 - \frac{x_0}{x_{r_i} q^{k_i}}\right) \right]. \quad (4.6)$$

Note that the product on the right hand side of (4.6) cancels all the factors in the denominator of Q that would be taken to zero by $E_{\mathbf{r}, \mathbf{k}}$, and if $k_i = 0$ for some i and $r_s > m$ then $Q(h \mid \mathbf{r}; \mathbf{k}) = 0$.

As a warm up, it is easy to check that $Q(h)$ is proper in x_0 with degree $-nh - m$. Thus applying Lemma 4.1 gives

$$\text{CT}_{x_0} Q(h) = \sum_{\substack{1 \leq r_{1,1} \leq m \\ 0 \leq k_{1,1} \leq h}} Q(h \mid r_{1,1}; k_{1,1}) + \sum_{\substack{m+1 \leq r_{2,1} \leq n \\ 1 \leq k_{2,1} \leq h}} Q(h \mid r_{2,1}; k_{2,1}). \quad (4.7)$$

Note that $1 - x_0/x_i$ is not a factor in the denominator of $Q(h)$ for $i > m$, so $Q(h \mid r_{2,1}; 0) = Q(h)(1 - x_0/x_{r_{2,1}})|_{x_0=x_{r_{2,1}}} = 0$. Thus we can rewrite (4.7) as

$$\text{CT}_{x_0} Q(h) = \sum_{\substack{1 \leq r_1 \leq n \\ 0 \leq k_1 \leq h}} Q(h \mid r_1; k_1). \quad (4.8)$$

This formula is compatible with the following lemma if we treat $Q(h) = Q(h \mid \emptyset; \emptyset)$.

Lemma 4.3. *The rational functions $Q(h \mid \mathbf{r}; \mathbf{k})$ have the following two properties:*

- (i) *If $0 \leq k_i \leq (s-1)k + b + \chi(s \geq n-l+1)$ for all i with $1 \leq i \leq s$, then $Q(h \mid \mathbf{r}; \mathbf{k}) = 0$.*
- (ii) *If $k_i > (s-1)k + b + \chi(s \geq n-l+1)$ for some i with $1 \leq i \leq s$ and $n > s$, then*

$${}_{x_{r_s}} \text{CT} Q(h \mid \mathbf{r}; \mathbf{k}) = \sum_{\substack{r_s < r_{s+1} \leq n \\ 0 \leq k_{s+1} \leq h}} Q(h \mid r_1, \dots, r_s, r_{s+1}; k_1, \dots, k_s, k_{s+1}). \quad (4.9)$$

Proof of property (i). By Lemma 4.2, either $0 \leq k_i \leq b$ for some i with $1 \leq i \leq s$, or $1 - k \leq k_j - k_i \leq k$ for some $i < j$, except only when $k_i = (s-i)k + b + 1$ for $i = 1, \dots, s$. Note that the exceptional case occurs only when $s \geq n-l+1$. For some i , if $0 \leq k_i \leq b$ and $r_s \leq m$, then $Q(h \mid \mathbf{r}; \mathbf{k})$ has the factor

$$E_{\mathbf{r}, \mathbf{k}} \left[\left(\frac{x_{r_i}}{x_0} \right)_{b+1+\chi(r_i \geq n-l+1)} \right] = \left(\frac{x_{r_s} q^{k_s - k_i}}{x_{r_s} q^{k_s}} \right)_{b+1+\chi(r_i \geq n-l+1)} = (q^{-k_i})_{b+1+\chi(r_i \geq n-l+1)} = 0.$$

If $0 \leq k_i \leq b$ and $r_s > m$ for some i , then $Q(h \mid \mathbf{r}; \mathbf{k}) = 0$ for $k_i = 0$ by the definition and for $1 \leq k_i \leq b$ it has the factor

$$E_{\mathbf{r}, \mathbf{k}} \left[\left(\frac{x_{r_i} q}{x_0} \right)_{b+\chi(r_i \geq n-l+1)} \right] = \left(\frac{x_{r_s} q^{1+k_s-k_i}}{x_{r_s} q^{k_s}} \right)_{b+\chi(r_i \geq n-l+1)} = (q^{1-k_i})_{b+\chi(r_i \geq n-l+1)} = 0.$$

If $1 - k \leq k_j - k_i \leq k$ where $i < j$, then $Q(h \mid \mathbf{r}; \mathbf{k})$ has the factor

$$E_{\mathbf{r}, \mathbf{k}} \left[\left(\frac{x_{r_i}}{x_{r_j}} \right)_k \left(\frac{x_{r_j}}{x_{r_i}} q \right)_k \right],$$

which is equal to

$$E_{\mathbf{r}, \mathbf{k}} \left[q^{\binom{k+1}{2}} \left(-\frac{x_{r_j}}{x_{r_j}} \right)^k \left(\frac{x_{r_i}}{x_{r_j}} q^{-k} \right)_{2k} \right] = q^{\binom{k+1}{2}} (-q^{k_i-k_j})^k (q^{k_j-k_i-k})_{2k} = 0.$$

If $k_i = (s-i)k + b + 1$ for $i = 1, \dots, s$, then $k_s = b + 1$. Thus $r_s \geq s \geq n-l+1$. Then if $r_s > m$, $Q(h \mid \mathbf{r}; \mathbf{k})$ has the factor

$$E_{\mathbf{r}, \mathbf{k}} \left[\left(\frac{x_{r_s} q}{x_0} \right)_{b+1} \right] = \left(\frac{x_{r_s} q}{x_{r_s} q^{k_s}} \right)_{b+1} = (q^{1-k_s})_{b+1} = (q^{-b})_{b+1} = 0,$$

and if $r_s \leq m$, then $m \geq n-l+1$ and $Q(h \mid \mathbf{r}; \mathbf{k})$ has the factor

$$E_{\mathbf{r}, \mathbf{k}} \left[\left(\frac{x_{r_s}}{x_0} \right)_{b+2} \right] = \left(\frac{x_{r_s}}{x_{r_s} q^{k_s}} \right)_{b+2} = (q^{-k_s})_{b+2} = (q^{-b-1})_{b+2} = 0.$$

Proof of property (ii). Note that since $h \geq k_i$ for all i and $h \in D_1 \cup D_2 \cup D_3$, the hypothesis implies that $h > sk - \chi(s < m)$.

We only show that $Q(h \mid \mathbf{r}; \mathbf{k})$ is proper in x_{r_s} so that Lemma 4.1 applies. The rest is the same as that in the proof of [7, Lemma 5.1]. To this end we write $Q(h \mid \mathbf{r}; \mathbf{k})$ as N/D , in which N (the numerator) is

$$E_{\mathbf{r}; \mathbf{k}} \left[\prod_{i=1}^m \left(\frac{x_i}{x_0} \right)_{b+1+\chi(i \geq n-l+1)} \prod_{i=m+1}^n \left(\frac{x_i q}{x_0} \right)_{b+\chi(i \geq n-l+1)} \cdot \prod_{\substack{1 \leq i, j \leq n \\ i \neq j}} \left(\frac{x_i}{x_j} q^{\chi(i > j)} \right)_k \right],$$

and D (the denominator) is

$$E_{\mathbf{r}; \mathbf{k}} \left[\prod_{i=1}^m \left(\frac{x_0}{x_i q^h} \right)_{h+1} \prod_{i=m+1}^n \left(\frac{x_0}{x_i q^h} \right)_h / \prod_{i=1}^s \left(1 - \frac{x_0}{x_{r_i} q^{k_i}} \right) \right]$$

Now let $R = \{r_0, r_1, \dots, r_s\}$. Then the degree in x_{r_s} of

$$E_{\mathbf{r}; \mathbf{k}} \left[\left(1 - \frac{x_i}{x_j} q^l \right) \right]$$

is 1 if $i \in R$ and $j \notin R$, and is 0 otherwise, as is easily seen by checking the four cases. Thus the part of N contributing to the degree in x_{r_s} is

$$E_{\mathbf{r}; \mathbf{k}} \left[\prod_{i=1}^s \prod_{j \neq r_0, \dots, r_s} \left(\frac{x_{r_i}}{x_j} q^{\chi(r_i > j)} \right)_k \right],$$

which has degree $(n-s)sk$, and the part of D contributing to the degree in x_{r_s} is

$$E_{\mathbf{r}; \mathbf{k}} \left[\prod_{\substack{j \neq r_0, \dots, r_s \\ j \leq m}} \left(\frac{x_0}{x_j q^h} \right)_{h+1} \prod_{\substack{j \neq r_0, \dots, r_s \\ j > m}} \left(\frac{x_0}{x_j q^h} \right)_h \right],$$

which has degree at least $(n-s)h + \chi(s < m)$.

Thus the total degree of $Q(h \mid \mathbf{r}; \mathbf{k})$ in x_{r_s} is at most $(n-s)(sk - h) - \chi(s < m) < 0$, so $Q(h \mid \mathbf{r}; \mathbf{k})$ is proper in x_{r_s} . \square

Now we are ready to prove the vanishing lemma.

Proof of the vanishing lemma. Recall that $\text{CT}_{\mathbf{x}} Q(h) = M'_n(-a, b, k, m, l; q)$. We prove by induction on $n-s$ that

$$\text{CT}_{\mathbf{x}} Q(h \mid \mathbf{r}; \mathbf{k}) = 0;$$

the lemma is the case $s = 0$. (Note that taking the constant term with respect to a variable that does not appear has no effect.) We may assume that $s \leq n$ and $0 < r_1 < \dots < r_s \leq n$, since otherwise $Q(h \mid \mathbf{r}; \mathbf{k})$ is not defined. If $s = n$ then r_i must equal i for $i = 1, \dots, n$ and thus $Q(h \mid \mathbf{r}; \mathbf{k}) = Q(h \mid 1, 2, \dots, n; k_1, k_2, \dots, k_n)$, which by property (i) of Lemma 4.3 is 0, since for each i , $k_i \leq h \leq (n-1)k + b + 1$. Now suppose that $0 \leq s < n$. Applying $\text{CT}_{\mathbf{x}}$ to both sides of (4.9) gives

$$\text{CT}_{\mathbf{x}} Q(h \mid \mathbf{r}; \mathbf{k}) = \sum_{\substack{r_s < r_{s+1} \leq n \\ 0 \leq k_{s+1} \leq h}} \text{CT}_{\mathbf{x}} Q(h \mid r_1, \dots, r_s, r_{s+1}; k_1, \dots, k_s, k_{s+1})$$

when property (ii) of Lemma 4.3 applies. Thus by Lemma 4.3, $\text{CT}_{\mathbf{x}} Q(h \mid \mathbf{r}; \mathbf{k})$ is either 0 or is a sum of terms, all of which are 0 by induction. \square

5 Proof for the extra point

We need the following lemma.

Lemma 5.1. *Assume Theorem 1.3 holds for smaller n . Let $h = (n - l - 1)k + b + 1$. If $0 \leq m, l < n$ and $k > b + 1$ then*

$$\text{CT}_x Q(h \mid 1, \dots, n - l; (n - l - 1)k + b + 1, \dots, b + 1) = M_n(-h, b, k, m, l; q).$$

Proof. We have to split into the following two cases.

Case 1: $m + l \leq n$. Then

$$Q(h \mid 1, \dots, n - l; h, h - k, \dots, b + 1) = D \cdot (A \cdot B) \cdot \prod_{n-l+1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k, \quad (5.1)$$

where

$$\begin{aligned} D &= \prod_{i=1}^m \frac{(q^{-(n-l-i)k-b-1})_{b+1}}{(q)_{(n-l-i)k+b+1} (q^{-(i-1)k})_{(i-1)k}} \prod_{i=m+1}^{n-l} \frac{(q^{-(n-l-i)k-b})_b}{(q)_{(n-l-i)k+b} (q^{-(i-1)k})_{(i-1)k}} \\ &\quad \times \prod_{1 \leq i < j \leq n-l} (-1)^k q^{(j-i)k^2 + \binom{k+1}{2}} (q^{-(j-i+1)k})_{2k}, \end{aligned} \quad (5.2)$$

$$\begin{aligned} A &= \prod_{i=n-l+1}^n \frac{\left(\frac{x_i}{x_{n-l}} q^{-b} \right)_{b+1}}{\left(\frac{x_{n-l}}{x_i} q^{-(n-l-1)k} \right)_{(n-l-1)k} \left(\frac{x_{n-l}}{x_i} \right)_{b+1}} \\ &= \prod_{i=n-l+1}^n \frac{\left(\frac{x_i}{x_{n-l}} q^{-b} \right)_{b+1}}{\left(\frac{x_{n-l}}{x_i} q^{-(n-l-1)k} \right)_{(n-l-1)k+b+1}}, \end{aligned} \quad (5.3)$$

and

$$\begin{aligned} B &= \prod_{\substack{1 \leq i \leq n-l \\ n-l+1 \leq j \leq n}} \left(\frac{x_{n-l}}{x_j} q^{-(n-l-i)k} \right)_k \left(\frac{x_j}{x_{n-l}} q^{(n-l-i)k+1} \right)_k \\ &= \prod_{j=n-l+1}^n \left(\frac{x_{n-l}}{x_j} q^{-(n-l-1)k} \right)_{(n-l)k} \left(\frac{x_j}{x_{n-l}} q \right)_{(n-l)k}. \end{aligned} \quad (5.4)$$

For $k > b + 1$, after cancelations and combinations, we obtain

$$\begin{aligned} &\text{CT}_x A \cdot B \cdot \prod_{n-l+1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k \\ &= \text{CT}_x \prod_{i=n-l+1}^n \left(\frac{x_{n-l}}{x_i} q^{b+1} \right)_{k-b-1} \left(\frac{x_i}{x_{n-l}} q^{-b} \right)_{(n-l)k+b+1} \prod_{n-l+1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k \\ &= \text{CT}_x \prod_{i=n-l+1}^n \left(\frac{x_{n-l}}{x_i} \right)_{k-b-1} \left(\frac{x_i}{x_{n-l}} q \right)_{(n-l)k+b+1} \prod_{n-l+1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k, \end{aligned} \quad (5.5)$$

where the last equality is obtained by making the substitution $x_{n-l} = x_{n-l}q^{-b-1}$. This is just $M'_l(k-b-1, (n-l)k+b+1, k, 0, 0; q)$. By Remark 2.4 (or the hypothesis), we obtain

$$\text{CT}_x Q(h) = D \cdot \prod_{i=0}^{l-1} \frac{(q)_{(n-l+i+1)k} (q)_{(i+1)k}}{(q)_{(i+1)k-b-1} (q)_{(n-l+i)k+b+1} (q)_k}, \quad (5.6)$$

which can be routinely checked to be equal to $M_n(-h, b, k, m, l; q)$.

Case 2: $m+l > n$. The computation is similar to but more complicated than case 1. Indeed we need the case 1 result in some sense. We omit some details for brevity. We have

$$Q(h \mid 1, \dots, n-l; h, h-k, \dots, b+1) = D' \cdot (A' \cdot B') \cdot \prod_{n-l+1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k, \quad (5.7)$$

where

$$D' = \prod_{i=1}^{n-l} \frac{(q^{-(n-l-i)k-b-1})_{b+1}}{(q)_{(n-l-i)k+b+1} (q^{-(i-1)k})_{(i-1)k}} \cdot \prod_{1 \leq i < j \leq n-l} (-1)^k q^{(j-i)k^2 + \binom{k+1}{2}} (q^{-(j-i+1)k})_{2k}, \quad (5.8)$$

and A' and B' are similar to A and B , with $A'B'$ simplifies as

$$\begin{aligned} A' \cdot B' &= \prod_{i=n-l+1}^m \left(\frac{x_{n-l}}{x_i} q^{b+2} \right)_{k-b-2} \left(\frac{x_i}{x_{n-l}} q^{-b-1} \right)_{(n-l)k+b+2} \\ &\quad \times \prod_{i=m+1}^n \left(\frac{x_{n-l}}{x_i} q^{b+1} \right)_{k-b-1} \left(\frac{x_i}{x_{n-l}} q^{-b} \right)_{(n-l)k+b+1}. \end{aligned} \quad (5.9)$$

A similar computation gives

$$\text{CT}_x A' \cdot B' \cdot \prod_{n-l+1 \leq i < j \leq n} \left(\frac{x_i}{x_j} \right)_k \left(\frac{x_j}{x_i} q \right)_k = M'_l(k-b-1, h+k, k, m-n+l, 0; q), \quad (5.10)$$

which is the constant term in (1.3) in case 1, and is known to be $M_l(k-b-1, h+k, k, m-n+l, 0; q)$ by the hypothesis. Then it only left to show that

$$D' \cdot \prod_{i=0}^{l-1} \frac{(q)_{(n-l+i+1)k} (q)_{(i+1)k}}{(q)_{(i+1)k-b-1-\chi(i < m-n+l)} (q)_{(n-l+i)k+b+1+\chi(i \geq n-m)} (q)_k} = M_n(-h, b, k, m, l; q),$$

which is routine. \square

Note that we can avoid using the induction hypothesis. The truth of Lemma 5.1 in case 1 results in the truth of Theorem 1.3 in case 1, which is needed in the case 2 of Lemma 5.1.

Now we are ready to deal with the extra point.

Proof of Lemma 2.5. As we discussed in (4.8), $\text{CT}_{x_0} Q(h)$ can be written as

$$\text{CT}_{x_0} Q(h) = \sum_{\substack{1 \leq r_1 \leq n \\ 0 \leq k_1 \leq h}} Q(h \mid r_1, k_1). \quad (5.11)$$

Iteratively apply Lemma 4.3 to each summand if applicable. Finally we get

$$\text{CT}_{\mathbf{x}} Q(h) = \text{CT}_{\mathbf{x}} \sum_{r_1, \dots, r_s, k_1, \dots, k_s} Q(h \mid r_1, \dots, r_s; k_1, \dots, k_s),$$

where the sum ranges over all r 's and k 's with $0 < r_1 < \dots < r_s \leq n, 0 \leq k_1, k_2, \dots, k_s \leq h$ such that Lemma 4.3 does not apply. Note that we may have different s .

Since $h = (n - l - 1)k + b + 1$ and $0 \leq k_i \leq h$, by Lemma 4.2, there leaves only one term for which Lemma 4.3 is not applicable. This term corresponds to $k_i = (n - l - i)k + b + 1$ for $i = 1, \dots, n - l$ and $r_i = i$ for $i = 1, \dots, n - l$. It follows that

$$\text{CT}_x Q(h) = \text{CT}_x Q(h \mid 1, \dots, n - l; (n - l - 1)k + b + 1, \dots, b + 1). \quad (5.12)$$

The lemma then follows from Lemma 5.1. \square

The extra point $h = (n - l - 1)k + b + 1$ in Lemma 2.5 is not easy to find. This h seems to be the only choice of the extra point for which it is not hard to show that $\text{CT}_x Q(h) = M_n(-h, b, k, m, l; q)$. Intuitively a desired extra point must be chosen from boundary values, i.e., values next to the vanishing points listed in (2.2).

Firstly, the boundary values $h = (n - l - 2)k + b + 1, (n - l - 3)k + b + 1, \dots, b + 1$ do not work. To see this, take $n = 3, m = l = 1$ for example. Then we can only get

$$\begin{aligned} Q(b + 1) = & (-1)^{b+1} q^{-\binom{b+2}{2}} \text{CT}_x \frac{(1/x_2)_{k-b-1} (x_2 q)_{b+k+1} (1/x_3)_{k-b-1} (x_3 q)_{b+k+1}}{1 - x_2 q^{b+1}} \cdot \left(\frac{x_2}{x_3}\right)_k \left(\frac{x_3}{x_2} q\right)_k \\ & + (-1)^b q^{-\binom{b+1}{2}} \text{CT}_x \frac{(q/x_1)_{k-b-1} (x_1)_{b+k+1} (1/x_3)_{k-b-1} (x_3 q)_{b+k+1}}{1 - 1/(x_1 q^{b+1})} \left(\frac{x_1}{x_3}\right)_k \left(\frac{x_3}{x_1} q\right)_k. \end{aligned}$$

Secondly, the boundary values $h = mk, (m + 1)k, \dots, (n - 1)k$ do not work either. But we obtain the following nice identity, which seems not have appeared in the literature.

Proposition 5.2.

$$\sum_{0 \leq t \leq k-b-1} \frac{(-1)^t q^{(n-1)kt + (b+1)t + \binom{t+1}{2}}}{(q)_{k-t-b-1} (q)_t} = \begin{bmatrix} nk \\ k - b - 1 \end{bmatrix}. \quad (5.13)$$

This identity is obtained through computing $Q(k)$ in the case $m = 1$ with general n and $l < n$. The computation is lengthy, and involves one application of the q -Dyson theorem and one application of Theorem 1.3. It might be worth considering other cases, which might lead to new combinatorial identities.

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